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ON MOTIONS ASYMPTOTIC TO TRIANGULAR POINTS OF LIBRATION OF THE RESTRICTED CIRCULAR THREE-BODY PROBLEM*

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The problem of the trajectories, asymptotic to the equilibrium positions of an autonomous Hamiltonian system with two degrees of freedom, is studied, in the case when the characteristic equation has pure imaginary roots. The results are used to find all the motions, asymptotic to the triangular points of libration, stable to a first approximation of the plane circular restricted three-body problem.

1. Formulation of the problem. The circular three-body problem admits of five exact particular solutions, namely the points of libration L_i $(i = 1, 2, \ldots, 5)$, for which the relative distances between the bodies are constant /1/. In stellar mechanics and its applications, there is great interest in the natural families of solutions of the equations of motion, for which the configuration formed by three moving bodies under the influence of forces of mutual gravitational attraction when there are no active control forces, tend asymptotically, as $t \to \pm\infty$, to configurations corresponding to points of libration.

A particularly important case for applications is the plane circular restricted three-body problem. The motions of a body of infinitesimal mass, asymptotic to the rectilinear points of libration L_1, L_2, L_3 of the problem, were studied in detail in /2-5/; the motions asymptotic to the triangular points of libration (TPL) L_4, L_5 for values of the parameter μ satisfying the condition $27\mu(1-\mu) > 1$, were studied in /4-7/ (if the sum of the masses of the main gravitating bodies is taken as unity, when μ is the mass of the lesser one).

The analysis in /2-7/ is based on the theory of asymptotic motions developed by Lyapunov and Poincaré. This theory gives the sufficient conditions for the existence of asymptotic motions and a constructive means of obtaining them in the form of series. One of the basic conditions for it to be applicable is that there exist in the linearized system of equations of the perturbed motion at least one non-zero characteristic number. This condition is satisfied in /2-7/.

If we have

$$0 < 27\mu(1-\mu) < 1 \tag{1.1}$$

the roots of the characteristic equation of the linearized system of the perturbed motion in the neighbourhood of the TPL are purely imaginary. Hence the characteristic numbers are zero for μ which satisfy (1.1) (the stability condition for L_4 and L_5 to a first approximation), and the Lyapunov and Poincaré theory of asymptotic motions is not applicable.

In the present paper we solve the problem of the motions, asymptotic to the TPL, of a body of infinitesimal mass in the plane circular restricted three-body problem for values of the parameter μ in the domain (1.1). The solution is based on our study in Sect.2 of the problem of the existence and construction of the solutions, asymptotic to the equilibrium position, in the autonomous Hamiltonian system with two degrees of freedom in the case when the characteristic equation has purely imaginary roots.

2. On the asymptotic motions of an autonomous Hamiltonian system with two degrees of freedom. Consider the system of differential equations

$$\frac{dq_i}{dt} = \frac{\partial H}{\partial p_i}, \quad \frac{dp_i}{dt} = -\frac{\partial H}{\partial q_i} \quad (i = 1, 2) \tag{2.1}$$

We shall assume that the origin $q_1 = q_2 = p_1 = p_2 = 0$ is a solution of this system, and that the Hamiltonian can be written as a convergent series in its neighbourhood

$$H = H_2 + H_3 + H_4 + \ldots + H_k + \ldots \tag{2.2}$$

where H_k is a form of degree k in q_1, q_2, p_1, p_2 with constant coefficients.

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System (2.1) has the energy integral $H(q_1, q_2, p_1, p_2) = h$. It is obvious that $h = 0$ in motions asymptotic to the origin, i.e., the asymptotic trajectories (if they exist) must lie at the zero energy level $H = 0$.

If the Hamiltonian (2.2) is of fixed sign, it can be zero only at $q_i = p_i = 0$ $(i = 1, 2)$, so that in this case there are no solutions of (2.1) which are asymptotic to the origin.

Now let H_2 (and hence H also) be of alternating sign, let the linearized system (2.1) with Hamiltonian H_2 be stable, and let the roots $\pm i\omega_1, \pm i\omega_2$ of the characteristic equation be purely imaginary and distinct $(\omega_1 > \omega_2 > 0)$. In this case, given a suitable choice of variables q_j, p_j $(j = 1, 2)$ (realized e.g., by the Birkhoff transformation /8/) the expansion (2.2) can be reduced /1/ to the following normal forms:

a) with third-order resonance $(\omega_1 = 2\omega_2)$

$$H = \omega_1 r_1 - \omega_2 r_2 + \chi_1 r_2 \sqrt{r_1} + O_4 \tag{2.3}$$

b) with fourth-order resonance $(\omega_1 = 3\omega_2)$

$$H = \omega_1 r_1 - \omega_2 r_2 + c_{20} r_1^2 + c_{11} r_1 r_2 + c_{02} r_2^2 + \chi_2 r_2 \sqrt{r_1 r_2} + O_5 \tag{2.4}$$

c) when there is no resonance up to and including the fourth order $(\omega_1 \neq \omega_2, 2\omega_2, 3\omega_2)$

$$H = \omega_1 r_1 - \omega_2 r_2 + c_{20} r_1^2 + c_{11} r_1 r_2 + c_{02} r_2^2 + O_5 \tag{2.5}$$

Here,

$$q_j = \sqrt{2r_j} \sin \varphi_j, \quad p_j = \sqrt{2r_j} \cos \varphi_j$$
$$\chi_j = \alpha_j \sin [\varphi_1 + (1 + j)\varphi_2] + \beta_j \cos [\varphi_1 + (1 + j)\varphi_2] \quad (j = 1, 2)$$

and $\alpha_j, \beta_j, c_{kl}$ are constants; O_4 and O_5 denote series in powers of q_j, p_j starting with terms of not lower than the fourth and fifth orders respectively.

Since motions asymptotic to the origin are only possible at the isoenergetic level $H = 0$, this is the level that we consider. Solving the equation $H = 0$ for r_2, we obtain:

a) with $\omega_1 = 2\omega_2$

$$r_2 = -K(r_1, \varphi_1; \varphi_2) \equiv 2r_1 + 2\omega_2^{-1}\chi_1 r_1 \sqrt{r_1} + K^{(2)}(r_1, \varphi_1; \varphi_2) \tag{2.6}$$

b) with $\omega_1 = 3\omega_2$

$$r_2 = -K(r_1, \varphi_1; \varphi_2) \equiv 3r_1 + \omega_2^{-1}[c_{20} + 3c_{11} + 9c_{02} + 3\sqrt{3}\chi_2] r_1^2 + K^{(5/2)}(r_1, \varphi_1; \varphi_2) \tag{2.7}$$

c) with $\omega_1 \neq \omega_2, 2\omega_2, 3\omega_2$

$$r_2 = -K(r_1, \varphi_1; \varphi_2) \equiv \frac{\omega_1}{\omega_2} r_1 + \frac{D}{\omega_2^3} r_1^2 + K_1^{(5/2)}(r_1, \varphi_1; \varphi_2) \tag{2.8}$$
$$D = c_{20}\omega_2^2 + c_{11}\omega_1\omega_2 + c_{02}\omega_1^2$$

The functions $K(r_1, \varphi_1; \varphi_2)$ in (2.6)-(2.8) are periodic in φ_1 and φ_2, while $K^{(l)}, K_1^{(l)}$ denote terms of not lower than order l in r_1.

The equations of motion at the level $H = 0$ (Whittaker's equations /9/) have a canonical form with Hamiltonian $K(r_1, \varphi_1; \varphi_2)$; the roles of momentum and coordinate are played by r_1 and φ_1 respectively, while the independent variable is φ_2.

It follows from the equations of motion corresponding to Hamiltonians (2.3)-(2.5) that

$$d\varphi_2/dt = -\omega_2 + \Phi(r_1, r_2, \varphi_1, \varphi_2) \tag{2.9}$$

where, as $r_1 + r_2 \to 0$, the function Φ tends to zero uniformly with respect to φ_1 and φ_2. In a sufficiently small neighbourhood of the origin, φ_2 is a monotonically decreasing function of time, so that, in the problem of the motions, asymptotic to the origin, φ_2 plays the role of $(-t)$.

If there are no resonances up to and including the fourth order and we have

$$D \neq 0 \tag{2.10}$$

or if there is the resonance $\omega_1 = 3\omega_2$ and we have

$$|c_{20} + 3c_{11} + 9c_{02}| > 3\sqrt{3(\alpha_2^2 + \beta_2^2)} \tag{2.11}$$

then the origin $r_1 = 0$ (of the plane with polar coordinates $\sqrt{2r_1}, \varphi_1$) is surrounded by invariance curves arranged arbitrarily close to the point $r_1 = 0$ [1, 10]. In these two cases, therefore, there are no trajectories, asymptotic to the origin; otherwise, the uniqueness of the solution of the Cauchy problem for differential Eqs.(2.1) would be violated.

Below we consider the problem of the asymptotic solutions in the case of resonance $\omega_1 = 2\omega_2$ with $\alpha_1^2 + \beta_1^2 \neq 0$, and also in the case of resonance $\omega_1 = 3\omega_2$, when inequality

(2.11) holds with the opposite sign. Notice that in these, as distinct from the previous two cases, the equilibrium position $q_j = p_j = 0$ $(j = 1, 2)$ of system (2.1) is unstable /1/.

Thus, let $\omega_1 = 2\omega_2$. The equations of motion at the level $H = 0$ will be canonical with Hamiltonian K given by (2.6). Assuming that $\alpha_1^2 + \beta_1^2 \neq 0$, we make the change of variables $r_1, \varphi_1 \to \rho, \theta$ in accordance with the relations

$$\varphi_1 = \frac{\pi}{2} + \sigma_1 - 2\varphi_2 - \theta, \quad r_1 = \frac{\omega_2^2}{\delta_1^2}\rho^2 \quad (\rho > 0) \tag{2.12}$$
$$\sin\sigma_1 = -\frac{\alpha_1}{\delta_1}, \quad \cos\sigma_1 = -\frac{\beta_1}{\delta_1}, \quad \delta_1 = \sqrt{\alpha_1^2 + \beta_1^2}$$

We can regard ρ, θ as polar coordinates in the Oxy plane:

$$x = \rho\cos\theta, \quad y = \rho\sin\theta \tag{2.13}$$

In the new variables the equations of motion become

$$\frac{d\theta}{d\varphi_2} = \rho F(\theta) + \Theta(\rho, \theta; \varphi_2), \quad \frac{d\rho}{d\varphi_2} = \rho^2 G(\theta) + R(\rho, \theta; \varphi_2) \tag{2.14}$$
$$(F(\theta) = -3\sin\theta, \ G(\theta) = \cos\theta)$$

where the functions Θ and R can be written as series, convergent in a neighbourhood of the origin, in powers of ρ with bounded coefficients, periodic in θ and φ_2, where the series start with terms of not lower than the second and third respectively.

We can apply to system (2.14) the results of /11-13/ concerning the behaviour of the trajectories of two differential equations in the neighbourhood of a singular point. The fact that the right-hand sides of the equations considered in /11-13/ do not contain the independent variable explicitly proves to be of no importance here, since, to apply these results, it suffices that, for some $\varepsilon > 0$, the functions Θ and R tend to zero as $\rho \to 0$ faster than $\rho^{1+\varepsilon}$ and $\rho^{2+\varepsilon}$ respectively, and that this convergence be uniform, not only with respect to the variable θ, but also with respect to φ_2. This condition is satisfied for system (2.14).

On repeating almost exactly the arguments of /11-13/, we find that the trajectories of system (2.14) can enter the origin only along the directions given by the angles $\theta = \theta_*$, where θ_* is a root of the equation $F(\theta) = 0$; if θ_* is a root of odd multiplicity and $GdF/d\theta$ is negative fpr $\theta = \theta_*$, then there is a unique integral curve entering the origin in the direction $\theta = \theta_*$.

Using the expressions for F and G, we find that system (2.14) has precisely two asymptotic trajectories, entering the origin: one at an angle $\theta_* = \pi$ as $\varphi_2 \to +\infty$, and the other at an angle $\theta_* = 0$ as $\varphi_2 \to -\infty$.

The behaviour of the trajectories of system (2.14) in the neighbourhood of the origin is shown in Fig.1. The direction of the arrows corresponds to increasing φ_2 (i.e., decreasing t).

The analyic form of the asymptotic trajectories can be found by passing in system (2.14) to Cartesian coordinates x, y in accordance with (2.13) and applying the results of /14/, Chapter 3, Sect.3 to the transformed system. We find that the solutions of system (2.14), corresponding to the asymptotic trajectories, can be written for sufficiently large values of $|\varphi_2|$ as series in powers of φ_2^{-1}, c, where c is an arbitrary constant. As $|\varphi_2| \to \infty$, x and y have order $|\varphi_2|^{-1}$.

Using the solutions $\rho(\varphi_2, c), \theta(\varphi_2, c)$ of system (2.14), we can obtain from (2.12) and (2.6) expressions for r_1, φ_1, and r_2 in terms of the variable φ_2 and the arbitrary constant c. On substituting these expressions into (2.9), we can then find the dependence of φ_2 on the time t by means of a quadrature. Thus, a further arbitrary constant appears. In short, the expressions for the asymptotic solutions inthe initial variables q_j, p_j $(j = 1, 2)$ contain two arbitrary constants.

To find the asymptotic solutions approximately, we use the equations of motion with Hamiltonian (2.3), where we discard terms of higher than the third order in q_j, p_j $(j = 1, 2)$. We then find that the solutions, asymptotic as $t \to \pm\infty$ to the origin $q_j = p_j = 0$ $(j = 1, 2)$, of system (2.1) can be written approximately as

$$q_1 = \pm\psi_1^{\pm}\cos(2\varkappa_1 + \sigma_1), \quad q_2 = -\sqrt{2}\psi_1^{\pm}\sin\varkappa_1 \tag{2.15}$$
$$p_1 = \mp\psi_1^{\pm}\sin(2\varkappa_1 + \sigma_1), \quad p_2 = \sqrt{2}\psi_1^{\pm}\cos\varkappa_1$$
$$\psi_1^{\pm} = \frac{\sqrt{2c_1}}{1 \pm \delta_1\sqrt{c_1}\,t}, \quad \varkappa_1 = \omega_2 t + c_2$$

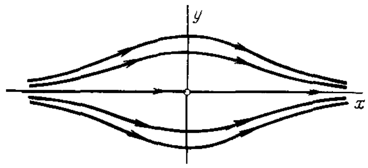

Fig.1

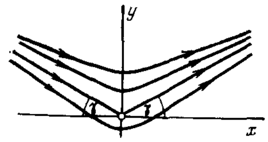

Fig.2

Here we take simultaneously either only the upper, or only the lower, signs; c_1, c_2 $(c_1 > 0)$ are arbitrary constants.

Now let there no no third-order resonance, but let us have fourth-order resonance $\omega_1 = 3\omega_2$ and inequality (2.11) with the reverse sign. We introduce into the canonical equations with the Hamiltonian K of (2.7) the new (non-canonical) variables in accordance with

$$\varphi_1 = \frac{\pi}{2} + \sigma_2 - 3\varphi_2 - \theta, \quad r_1 = \frac{\omega_2}{6\sqrt{3}\,\delta_2}\rho^2 \quad (\rho > 0)$$
$$\sin\sigma_2 = -\alpha_2/\delta_2, \ \cos\sigma_2 = -\beta_2/\delta_2, \ \delta_2 = \sqrt{\alpha_2^2 + \beta_2^2}$$

In the new variables the equations of motion at energy level $H = 0$ will be

$$\frac{d\theta}{d\varphi_2} = \rho^2 F(\theta) + \Theta(\rho, \theta; \varphi_2), \quad \frac{d\rho}{d\varphi_2} = \rho^3 G(\theta) + R(\rho, \theta; \varphi_2) \tag{2.16}$$
$$\left(F(\theta) = \sin\gamma - \sin\theta, \ G(\theta) = \frac{1}{4}\cos\theta, \quad \gamma = \arcsin\frac{c_{20} + 3c_{11} + 9c_{02}}{3\sqrt{3}\,\delta_2}\right)$$

where Θ and R are analytic functions of ρ in the neighbourhood of the origin, whose series expansions have bounded coefficients, periodic in θ and φ_2 and start with terms of not lower than the third and fourth orders respectively.

In the same way as in the above case of third-order resonance, we can apply the results of /11-14/ to system (2.16). We find that it has just two asymptotic trajectories, entering the orign: one at an angle $\theta_* = \pi - \gamma$ as $\varphi_2 \to +\infty$, and the other at an angle $\theta_* = \gamma$ as $\varphi_2 \to -\infty$. As $|\varphi_2| \to \infty$, x and y have order $|\varphi_2|^{-1/2}$. The behaviour of the trajectories in the neighbourhood of the origin is shown in Fig.2 (we take $\gamma > 0$).

In the initial variables t, q_j, p_j $(j = 1, 2)$ we have two two-parameter asymptotic solutions, tending to the origin $q_j = p_j = 0$ $(j = 1, 2)$. We obtain their approximate expressions from the equations of motion with Hamiltonian (2.4), in which we neglect terms of the fourth order and above in q_j, p_j $(j = 1, 2)$. We have

$$q_1 = \pm\psi_2^{\pm}\cos(3\varkappa_2 + \sigma_2 \mp \gamma), \ q_2 = -\sqrt{3}\psi_2^{\pm}\sin\varkappa_2 \tag{2.17}$$
$$p_1 = \mp\psi_2^{\pm}\sin(3\varkappa_2 + \sigma_2 \mp \gamma), \quad p_2 = \sqrt{3}\psi_2^{\pm}\cos\varkappa_2$$

$$\psi_2^{\pm} = \frac{\sqrt{2c_1}}{\sqrt{1 \pm \varkappa c_1 t}}, \quad 3\varkappa_2 = \omega_1 t \pm \sigma\ln(1 \pm \varkappa c_1 t) + c_2$$
$$\varkappa = 3\sqrt{3}\,\delta_2\cos\gamma, \quad \sigma = \sqrt{3}\ \frac{c_{20} + c_{11} - 3c_{02}}{6\delta_2\cos\gamma}$$

Here, we take simultaneously either only the upper, or only the lower, signs; c_1, c_2 $(c_1 > 0)$ are arbitrary constants.

Combining our results, we obtain the following theorem on the solutions of system (2.1), asymptotic to the origin as $t \to \pm\infty$

Theorem 1. If the Hamiltonian H is of fixed sign, there are no asymptotic solutions. If H is of alternating sign, and the roots of the characteristic equation of the linearized system (2.1) are purely imaginary $\pm i\omega_1, \pm i\omega_2$, then 1) when there are no resonances up to and including the fourth order $(\omega_1 \neq \omega_2, \ \omega_1 \neq 2\omega_2, \ \omega_1 \neq 3\omega_2)$ and we have

$$c_{20}\omega_2^2 + c_{11}\omega_1\omega_2 + c_{02}\omega_1^2 \neq 0,$$

or if there is a fourth-order resonance $\omega_1 = 3\omega_2$ and we have

$$|c_{20} + 3c_{11} + 9c_{02}| > 3\sqrt{3(\alpha_2^2 + \beta_2^2)} \tag{2.18}$$

there are no asymptotic solutions; 2) if there is third-order resonance $\omega_1 = 2\omega_2$ and we have

$$\alpha_1^2 + \beta_1^2 \neq 0,$$

or if we have fourth-order resonance $\omega_1 = 3\omega_2$ and we have the reverse inequalities to (2.18), there are just two two-parameter asymptotic solutions as described above, and there are no other asymptotic solutions.

3. Motions asymptotic to triangular points of libration (TPL). Let ξ, η be the coordinates of a body of infinitesimal mass in the Cartesian coordinate system $L_4\xi\eta$ with origin at the libration point L_4; the $L_4\xi$ axis is in the direction of the line from the body of greater finite mass to the body of lesser finite mass, while the direction of shortest rotation from $L_4\xi$ to $L_4\eta$ is the same as the direction of rotation of the bodies of finite mass. The length and time measurement units are such that the distance between the finite masses and their angular velocity of rotation along the circular orbits about the common centre of mass are unity.

For values of the parameter μ of domain (1.1), the Hamiltonian (2.2) of the plane circular restricted three-body problem is of alternating sign, and the roots of the charactersitic equation of system (2.1), linearized close to L_4, are purely imaginary and distinct: $\pm i\omega_1$, $\pm i\omega_2$ $(\omega_1 > \omega_2 > 0)$ /1/. For the two values of μ of domain (1.1):

$$\mu_1 = {}^1/_2 - \sqrt{1833}/90, \quad \mu_2 = {}^1/_2 - \sqrt{213}/30 \tag{3.1}$$

we have third-order $(\omega_1 = 2\omega_2)$ and fourth-order $(\omega_1 = 3\omega_2)$ resonance respectively.

In domain (1.1) the TPL are Lyapunov stable for all μ except μ_1 and μ_2, at which we have instability /1/. On the basis of this result and the symmetry properties of the equations of motion, it was shown in /4/ that, for μ of (1.1) not equal to values (3.1), there are no motions asymptotic to TPL in the circular restricted three-body problem. The same conclusion (without using the symmetry properties) can be obtained by using Theorem 1. Admittedly, Theorem 1 then has to be generalized to the case when condition (2.10) is violated (which is possible in domain (1.1) /1/). This generalization is similar to that given in Chapter 4 of the book /1/ of the Arnol'd-Mozer theorem on the stability of the equilibrium positions of an autonomous Hamiltonian system with two degrees of freedom.

On the basis of Theorem 1, we consider the problem of the motions, asymptotic to the TPL, of a body of infinitesimal mass at the resonance values (3.1) of the parameter μ. Using the approximate expressions (2.15) and (2.17) of the asymptotic solutions, and the expressions for linear normalization and the numerical values of the coefficients of normal forms (2.3) and (2.4) of the Hamiltonian (/1/, Chapter 7, Sect.4), we find that, at third-order resonance $\omega_1 = 2\omega_2$ $(\mu = \mu_1)$, the motions of the body of infinitesimal mass are given by

$$\begin{gathered}\xi = \psi\,(-4{,}273 \sin\varphi \pm 2.326 \sin 2\varphi \mp 0.524 \cos 2\varphi) + \ldots \\ \eta = \psi\,(2.156 \sin\varphi - 1.560 \cos\varphi \mp 0.635 \sin 2\varphi \pm \\ 1.576 \cos 2\varphi) + \ldots \\ \psi = \frac{\alpha}{1 \pm 0.958\alpha t}, \quad \varphi = \frac{\sqrt{5}}{5} t + \beta\end{gathered} \tag{3.2}$$

Here, the upper sign refers to the motion, asymptotic to L_4 as $t \to +\infty$, and the lower, as $t \to -\infty$.

Similarly, at fourth-order resonance $\omega_1 = 3\omega_2$ $(\mu = \mu_2)$, for the motions asymptotic to L_4 as $t \to +\infty$, we have

$$\begin{gathered}\xi = \psi\,(-7.466 \sin\varphi - 1.411 \sin 3\varphi + 2.512 \cos 3\varphi) + \ldots \\ \eta = \psi\,(4.015 \sin\varphi - 2.009 \cos\varphi - 0.947 \sin 3\varphi - \\ 1.858 \cos 3\varphi) + \ldots \\ \psi = \frac{\alpha}{\sqrt{1 + 22.904\alpha^2 t}}, \quad \varphi = \frac{\sqrt{10}}{10} t - 0.061 \ln(1 + 22.904\alpha^2 t) + \beta\end{gathered} \tag{3.3}$$

and for the motions, asymptotic to L_4 as $t \to -\infty$

$$\begin{gathered}\xi = \psi\,(-7.466 \sin\varphi + 0.434 \sin 3\varphi - 2.848 \cos 3\varphi) + \ldots \\ \eta = \psi\,(4.015 \sin\varphi - 2.009 \cos\varphi + 1.541 \sin 3\varphi + \\ 1.404 \cos 3\varphi) + \ldots \\ \psi = \frac{\alpha}{\sqrt{1 - 22.904\alpha^2 t}}, \quad \varphi = \frac{\sqrt{10}}{10} t + 0.061 \ln(1 - 22.904\alpha^2 t) + \beta\end{gathered} \tag{3.4}$$

In (3.2-(3,4), α and β $(\alpha > 0)$ are arbitrary constants, while the dots denote terms of order α^2 of higher.

We have thus proved:

Theorem 2. In the plane restricted circular three-body problem, for values of the parameter μ from domain (1.1), trajectories asymptotic to the triangular libration point L_4 exist only when $\mu = \mu_1$ and $\mu = \mu_2$. These trajectories are given by (3.2)-(3.4); there are no other asymptotic trajectories to L_4.

The asymptotic trajectories (3.2)-(3.4) twist in a spiral fashion at the origin, i.e., at the libration point L_4. The trajecories asymptotic to L_5 can be found from (3.2)-(3.4) by using the symmetry properties of the equations of motion given in /4/.

REFERENCES

1. MARKEYEV A.P., Libration points in celestial mechanics and space dynamics (Tochki libratsiya v nebesnoi mekhanike i kosmodinamke), Nauka, Moscow, 1978.
2. LUK'YANOV L.G., Some asymptotic solutions in the restricted three-body problem, Celest. Mech., 13, 4, 1976.
3. LUK'YANOV L.G., A study of asymptotic solutions in the vicinity of collinear libration points of the restricted three-body problem, Celest. Mech., 15, 4, 1977.
4. LUK'YANOV L.G., On the plane shape and symmetry properties of the asymptotic solutions in the restricted three-body problem, Astron. Zh., 55, 1, 1978.
5. SEBEKHEI V., Theory of orbits (Teoriya orbit), Nauka, Moscow, 1982.
6. DANBY J., Two notes on the Copenhagen problem, Celest. Mech., 33, 3, 1984.
7. HENRARD J., Proof of a conjecture of E. Strömgren, Celest. Mech., 7, 4, 1973.
8. BIRKHOFF G.D., Dynamical Systems, Am. Math., 1966.
9. GANTMAKHER F.R., Lectures on Analytical Mechanics (Lektsii po analiticheskoi mekanike), Nauka, Moscow, 1966.
10. MOSER YU., Lectures on Hamiltonian Systems, Mir, Moscow, 1973.
11. FORSTER H., Über das Verhalten der Integralkurven einer gewöhnlichen Differentialgleichung erster Ordnung in der Umgebung eines singulären Punktes, Math. Z., 43, 2, 1937.
12. LONN E.R., Über singuläre Punkte gewöhnlicher Differentialgleichungen, Math. Z., 44, 4, 1938.
13. NEMITSKII V.V. and STEPANOV V.V., Qualtitative Theory of Differential Equations (Kachestvennaya teoriya differentsialn'nykh urvavnenii), Gostekhizdat, Moscow-Leningrad, 1949.
14. ZUBOV V.I., Stability of motion (Ustoichivost' dvizheniya), Vyssh. Shkola, Moscow, 1973.

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